The existence of closed 3-forms of $\tilde{G}_2$-type on 7-manifolds*

Hông-Vân Lê

Abstract

In this note we construct a first example of a closed 3-form of $\tilde{G}_2$-type on $S^3 \times S^4$. We prove that $S^3 \times S^4$ does not admit a homogeneous 3-form of $\tilde{G}_2$-type. Thus our example is a first example of a closed 3-form of $\tilde{G}_2$-type on a compact 7-manifold which is not stably homogeneous.

Key words: closed 3-forms, $\tilde{G}_2$-forms.

1 Introduction.

Let $\Lambda^k(V^n)^*$ be the space of k-linear anti-symmetric forms on a given linear space V^n. For each $\omega \in \Lambda^k(V^n)^*$ we denote by I_ω the linear map

$$I_\omega : V^n \to \Lambda^{k-1}(V^n)^*,\ x \mapsto (x\rfloor\omega) := \omega(x, \cdots).$$

A k-form ω is called **multi-symplectic**, if I_ω is a monomorphism. Two k-forms are equivalent, if they are in the same orbit of the action of $Gl(V^n)$ on $\Lambda^k(V^n)^*$.

The classification of multi-symplectic 3-forms on $\mathbb{R}^7$ has been done by Bures and Vanzura [1]. There are together 8 types of these forms, among them there are two stable forms of G_2-type and $\tilde{G}_2$-type. More precisely the orbits of these stable 3-forms under the action of $Gl(V^7)$ are open sets in $\Lambda^3(V^7)^*$ and their corresponding isotropy groups are the compact group G_2 and its dual non-compact group $\tilde{G}_2$.

There are many known results on 7-manifolds admitting a (closed) 3-form of G_2-type, see e.g. [2],[3], [6],[7], [9], [10], [11]. Manifolds which admit a (closed) 3-form of $\tilde{G}_2$-type are less known. In particular, known examples of closed 3-forms of $\tilde{G}_2$-type on compact 7-manifolds up to now are homogeneous examples or obtaining from those by adding a small closed 3-form. We call such a closed 3-form of $\tilde{G}_2$-type stably homogeneous.

In this note we construct a first example of a closed 3-form ω^3 of $\tilde{G}_2$-type on a manifold $X^7 = S^3 \times S^4$ by identifying X^7 with a submanifold of the group $SU(3)$ provided with the Cartan 3-form (Theorem 2.1.) In section 3 we prove that the codimension of the action of the full automorphism group of (X^7, ω^3) on X^7 is 1 (Theorem 3.1), thus (X^7, ω^3) is not homogeneous. Moreover we prove that

*MSC: 53C10, 53C42

X^7 admits no homogeneous $\tilde{G}_2$-structure (Proposition 3.2). So our example is a $\tilde{G}_2$-structure on $S^3 \times S^4$ with maximal symmetrie.

This note also contains an appendix which gives a necessary and sufficient condition for a closed 7-manifold to admit a $\tilde{G}_2$-structure. As a corollary we obtain many examples of open 7-manifolds admitting a closed 3-form of $\tilde{G}_2$-type.

2 New example of a closed 3-form of $\tilde{G}_2$-type on $S^3 \times S^4$.

On each semi-simple Lie group G there exists a natural bi-invariant 3-form ϕ^3 which is defined at the Lie algebra $g = T_eG$ as follows

$$\phi^3(X, Y, Z) =< X, [Y, Z] >,$$

where $<,>$ denotes the Killing form on g. This 3-form ϕ^3 is also called the Cartan 3-form.

We claim that the Cartan 3-form ω^3 is multi-symplectic. To show the injectivity of the linear map I_{ϕ^3} we notice that if $X \in \ker I_{\phi^3}$, then

$$< X, [Y, Z] >= 0 \text{ for all } Y, Z \in g.$$

But this condition is incompatible with the semi-simplicity of g.

Let us consider the group $G = SU(3)$. For each $1 \le i \le j \le 3$ let $g_{ij}(g)$ be the complex function on $SU(3)$ induced from the standard unitary representation ρ of $SU(3)$ on $\mathbb{C}^3$: $g_{ij}(g) :=< \rho(g)\circ e_i, \bar{e}_j >$. Here $\{e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)\}$ is a unitary basis of $\mathbb{C}^3$. Let us denote by X^7 the co-dimension 1 subset in $SU(3)$ which is defined by the equation $Im(g_{11}(g)) = 0$.

2.1. Theorem. *The subset X^7 is diffeomorphic to the manifold $S^3 \times S^4$. Moreover X^7 is provided with a closed 3-form of $\tilde{G}_2$-type which is the restriction of ϕ^3 to X^7.*

Proof. Let $SU(2)$ be the subgroup in $SU(3)$ consisting of all $g \in SU(3)$ such that $\rho(g) \circ e_1 = e_1$. We denote by π the natural projection

$$\pi : SU(3) \to SU(3)/SU(2).$$

We identify $SU(3)/SU(2)$ with the sphere $S^5 \subset \mathbb{C}^3$ using the standard representation ρ of $SU(3)$ on $\mathbb{C}^3$. Namely we set

$$\tilde{\rho}(g \cdot SU(2)) := \rho(g) \circ e_1.$$

We denote by Π the composition $\tilde{\rho} \circ \pi : SU(3) \to SU(3)/SU(2) \to S^5$. Let $S^4 \subset S^5$ be the geodesic sphere which consists of points $v \in S^5$ such that $Im\, e^1(v) = 0$. Here $\{e^i, i = 1, 2, 3\}$ are the complex 1-forms on $\mathbb{C}^3$ which are dual to $\{e_i\}$. The pre-image $\Pi^{-1}(S^4)$ consists of all $g \in SU(3)$ such that

$$Im\, e^1(\rho(g) \circ e_1) = 0.$$

$$\iff Im\,(g_{11}) = 0.$$

So X^7 is $SU(2)$-fibration over S^4. But this fibration is the restriction of the $SU(2)$-fibration $\Pi^{-1}(D^5)$ over the half-sphere D^5 to the boundary $\partial D^5 = S^4$. Hence X^7 is a trivial $SU(2)$-fibration. This proves the first statement of Theorem 2.2.

Let us denote by $SO(2)^1$ the orthogonal group of the real subspace $\mathbb{R}^2 \subset \mathbb{C}^3$ such that $\mathbb{R}^2$ is the span of e_1 and e_2 over $\mathbb{R}$. Clearly $SO(2)^1$ is a subgroup of $SU(3)$.

We denote by $m_L(g)$ (resp. $m_R(g)$) the left multiplication (resp. the right multiplication) by an element $g \in SU(3)$.

2.2. Lemma. *X^7 is invariant under the action of $m_L(SU(2)) \cdot m_R(SU(2))$. For each $v \in S^4$ there exist an element $\alpha \in SO(2)^1$ and an element $g \in SU(2)$ such that $\Pi(g \cdot \alpha) = v$. Consequently for any point $x \in X^7$ there are $g_1, g_2 \in SU(2)$ and $\alpha \in SO(2)^1$ such that*

$$x = g_1 \cdot \alpha \cdot g_2, \tag{2.2.1}$$

Proof. Using the identification that $X^7 = \Pi^{-1}(S^4)$ we note that the orbits of $m_R(SU(2))$-action on X^7 are the fibers $\Pi^{-1}(v)$. Now the first statement follows by straightforward calculations. Let $v = (\cos\alpha, z_2, z_3) \in S^4$, where $z_i \in \mathbb{C}$. We choose $\alpha \in SO(2)^1$ so that

$$\rho(\alpha) \circ e_1 = (\cos\alpha, \sin\alpha) \in \mathbb{R}^2.$$

Clearly α is defined by v uniquely up to sign $\pm$. We set

$$w := (\sin\alpha, 0) \in \mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}} .$$

We notice that

$$|z_2|^2 + |z_3|^2 = \sin^2\alpha.$$

Since $SU(2)$ acts transitively on the sphere S^3 of radius $|\sin\alpha|$ in $\mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}}$, there exists an element $g \in SU(2)$ such that $\rho(g) \circ w = (z_2, z_3)$. Clearly

$$\Pi(g \cdot \alpha) = v.$$

This proves the second statement. The last statement of Lemma 2.2 follows from the second statement and the fact that $X^7 = \Pi^{-1}(S^4)$ □

Continuation of the proof of the second part of Theorem 2.1. Let us write here a canonical expression of a 3-form ω^3 of $\tilde{G}_2$-type (see e.g. [2], [1])

$$\omega^3 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_3. \tag{2.3}$$

Here α_i are 2-forms on V^7 which can be written as

$$\alpha_1 = y_1 \wedge y_2 + y_3 \wedge y_4,\ \alpha_2 = y_1 \wedge y_3 - y_2 \wedge y_4,\ \alpha_3 = y_1 \wedge y_4 + y_2 \wedge y_3$$

and $(\theta_1, \theta_2, \theta_3, y_1, y_2, y_3, y_4)$ is an oriented basis of $(V^7)^*$.

Using Lemma 2.2 we reduce the proof of the second statement of Theorem 2.1 to verifying that the value of $\phi^3_{|X^7}$ at any point $\alpha \in SO(2)^1 \subset X^7$ is a 3-form of $\tilde{G}_2$-type. First we shall compute that

value at point $e \in SO(2)^1 \subset X^7$ and then we shall compute that value at any point $\alpha \in SO(2)^1$.

Step 1. We shall use the Killing metric to identify the Lie algebra $su(3)$ with its co-algebra g. In what follows we shall not distinguish co-vectors and vectors, poly-vector and exterior forms on $su(3)$. Clearly we have

$$T_eX^7 = \{v \in su(3) : Im\, g_{11}(v) = 0\}.$$

Now we identify $gl(\mathbb{C}^3)$ with $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ and we denote by e_{ij} the element of $gl(\mathbb{C}^3)$ of the form $e_i \otimes (e_j)^*$. Let δ_i be the 1-forms in T_eX^7 which are defined as follows:

$$\delta_1 = \frac{i}{\sqrt{2}}(e_{22} - e_{33}), \delta_2 = \frac{1}{\sqrt{2}}(e_{23} - e_{32}), \delta_3 = \frac{i}{\sqrt{2}}(e_{23} + e_{32}).$$

Furthermore, ω_i are 2-forms on T_eX^7 which have the following expressions:

$$2\omega_1 = -(e_{12} - e_{21}) \wedge i(e_{12} + e_{21}) + (e_{13} - e_{31}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_2 = -(e_{12} - e_{21}) \wedge (e_{13} - e_{31}) - i(e_{12} + e_{21}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_3 = -(e_{12} - e_{21}) \wedge i(e_{13} + e_{31}) + i(e_{12} + e_{21}) \wedge (e_{13} - e_{31}).$$

2.4. Lemma. *The restriction of the Cartan 3-form to X^7 at the point e is*

(2.4.1) $$\phi^3_{|T_eX^7} = \sqrt{2}\delta_1 \wedge \delta_2 \wedge \delta_3 + \frac{1}{\sqrt{2}}\omega_1 \wedge \delta_1 + \frac{1}{\sqrt{2}}\omega_2 \wedge \delta_2 + \frac{1}{\sqrt{2}}\omega_3 \wedge \delta_3,$$

Proof. We set

$$f_1 := \frac{e_{12} - e_{21}}{\sqrt{2}},\ f_2 := \frac{i(e_{12} + e_{21})}{\sqrt{2}}$$

$$f_3 := \frac{e_{13} - e_{31}}{\sqrt{2}},\ f4 := \frac{i(e_{13} + e_{31})}{\sqrt{2}}$$

Then $(\delta_1, \delta_2, \delta_3, f_1, f_2, f_3, f_4)$ form an ortho-normal basis of T_eX^7 w.r.t. to the restriction of the Killing metric to X^7. Next we observe that $(\delta_1, \delta_2, \delta_3)$ form a $su(2)$-algebra. Furthermore we have

(2.5.1) $$[\delta_2, \delta_3] = \sqrt{2}\delta_1, [\delta_1, \delta_2] = \sqrt{2}\delta_3, [\delta_1, \delta_3] = -\sqrt{2}\delta_2.$$

(2.5.2) $$[f_1, f_2] = i(e_{11} - e_{22}), [f_1; f_3] = -\frac{1}{\sqrt{2}}\delta_2, [f_1, f_4] = -\frac{1}{\sqrt{2}\delta_3}.$$

(2.5.3) $$[f_2, f_3] = \frac{1}{\sqrt{2}}\delta_3, [f_2, f_4] = -\frac{1}{\sqrt{2}}\delta_2.$$

(2.5.4) $$[f_3, f_4] = i(e_{11} - e_{33}).$$

Using (2.5.1) - (2.5.4) we compute all the values $\phi^3(X_1, X_2, X_3)$ easily, where either (X_1, X_2) are (δ_i, δ_j) or (X_1, X_2) are (f_i, f_j) for some (i, j) and X_3 is one of the basic vectors (δ_i, f_i). In this way we get the equality (2.4.1). $\square$

Now compare (2.4.1) with (2.3) we observe that these two 3-forms are $Gl(R^7)$ equivalent (e.g. by rescaling δ_i with factor (1/2)). This proves that $\phi^3_{|T_eX^7}$ is a 3-form of $\tilde{G}_2$-type. This completes the first step.

Step 2. Using step 1 it suffices to show that

$$D\, m_L(\alpha^{-1})(T_\alpha X^7) = T_e X^7 \tag{2.6}$$

for any $\alpha \in SO(2)^1 \subset X^7$, $\alpha \neq e$.

Since $X^7 \supset \alpha \cdot SU(2)$, we have

$$su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.7}$$

Denote by $SO(3)$ the standard orthogonal group of $\mathbb{R}^3 \subset \mathbb{C}^3$. Since $\alpha \in SO(3) \subset X^7$, we have $D\, m_L(\alpha^{-1})(T_\alpha SO(3)) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7)$. In particular we have

$$< (e_{12} - e_{21}), (e_{13} - e_{31}) >_{\otimes\mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.8}$$

Since $SU(2) \cdot \alpha \subset X^7$, we have

$$Ad(\alpha^{-1}) su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.9}$$

Using the formula

$$Ad(\alpha^{-1}) = \exp(-ad(t \cdot \frac{e_{12} - e_{21}}{\sqrt{2}})),\ t \not\equiv 0$$

we get immediately from (2.7), (2.8), (2.9) the following inclusion

$$< i(e_{12} + e_{21}), i(e_{13} + e_{31}) >_{\otimes\mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7))$$

which together with (2.7), (2.8) imply the desired equality (2.6).

This completes the proof of Theorem 2.1. $\square$

3 The orbits of the action of the automorphism group of (X^7, ω^3) on X^7.

Denote by $Aut(X^7, \omega^3)$ the full automorphism group of the manifold X^7 equipped by the 3-form ω^3 constructed in Theorem 2.1. In this section we shall prove the following

3.1. Theorem. *The co-dimension of the $Aut(X^7, \omega^3)$-action on X^7 is equal to 1.*

Lemma 2.2 implies that the group $SU(2) \times SU(2)$ is a subgroup of the automorphism group $Aut(X^7, \omega^3)$. Further taking into account (2.2.1) we note that the dimension of a generic orbit of the $Aut(X^7, \omega^3)$-action on X^7 is at least 6. Hence Theorem 3.1 is a corollary of the following Proposition, which implies that our manifold (X^7, ω^3) is not stably homogeneous.

3.2. Proposition. *Let G be a Lie group which acts transitively on X^7. Then there does not exist a G-invariant 3-form of $\tilde{G}_2$-type on X^7.*

Proof of Proposition 3.2. Since $S^3 \times S^4$ is connected, we can assume that G is connected, because the identity component of G acts also transitively on $S^3 \times S^4$. Denote by $\bar{G}$ the simply connected covering of G. Then $\bar{G}$ acts transitively and almost effectively on X^7. We shall prove the following

3.3. Proposition. *Let X^n be a compact connected space with $\pi_1(X^n) = 0 = \pi_2(X^n)$. Suppose that $\bar{G}$ is a connected and simply connected group which acts transitively X^n. Let $\bar{G}_{s,u}$ be a maximal compact group of $\bar{G}$. Then $\bar{G}_{s,u}$ acts transitively on X^n.*

Proof. We denote by $\bar{H}$ the isotropy of the action of $\bar{G}$ on X^n. Since X^n is simply connected, the subgroup $\bar{H}$ is connected. From the homotopy exact sequence

$$\pi_2(X^n) = 0 \to \pi_1(\bar{H}) \to \pi_1(\bar{G}) = 0$$

we obtain that $\bar{H}$is also simply connected. Using the Levi-Maltsev decomposition theorem we write

$$\bar{G} = \bar{G}_s \times V^k, \ \bar{H} = \bar{H}_s \times V^r,$$

where $\bar{G}_s$ is a semisimple Lie subgroup of $\bar{G}$, V^k is a solvable normal subgroup of $\bar{G}$ and $\bar{H}_s$ is semisimple Lie subgroup of $\bar{H}$ and V^r is a solvable normal subgroup of $\bar{H}$. Moreover we can assume that $\bar{H}_s$ is a subgroup of $\bar{G}_s$ after applying an inner automorphism group of $\bar{G}$. Using the fbration

$$V^r \to \bar{G}/\bar{H} \to \bar{G}/\bar{H}_s$$

we conclude that the quotient space $\bar{G}/\bar{H}_s$ is homotopic to $\bar{G}/\bar{H}$. Further using the fibration $V^k \to \bar{G}/\bar{H}_s \to \bar{G}_s/\bar{H}_s$ we conclude that $\bar{G}_s/\bar{H}_s$ has the same homotopy type of X^7.

Denote by h_s the Lie algebra of $\bar{H}_s$ and by $h_{s,u}$ the maximal compact Lie sub-algebra of the semi-simple Lie algebra h_s. The Lie subalgebra $h_{s,u}$ is the Lie algebra of a maximal compact Lie group $\bar{H}_{s,u}$ of $\bar{H}_s$. We can also assume that $h_{s,u}$ is a subalgebra of a maximal compact Lie subalgebra $g_{s,u}$ of g_s. Denote by $\bar{G}_{s,u}$ the maximal compact subgroup of $\bar{G}_s$ whose Lie algebra is $g_{s,u}$. Using the Iwasawa decomposition we conclude that the quotient space $\bar{G}_{s,u}/\bar{H}_{s,u}$ has the same homotopy type of X^n. Since $\bar{G}_{s,u}$ and $\bar{H}_{s,u}$ are compact and using

$$H_{i+1}(X^n, \mathbb{Z}) = 0, \text{ if } i \geq n, \ H_n(X^n, \mathbb{Z}) = \mathbb{Z},$$

we obtain that $\dim \bar{G}_{s,u} = \dim \bar{H}_{s,u} + n$. Thus to show that the action of $\bar{G}_{s,u}$ on X^n is transitively, it suffces to prove

$$g_{s,u} \cap h = h_{s,u}, \tag{3.4}$$

where h is the Lie algebra of $\bar{H}$. We have the Iwasawa and Levi-Maltsev decomposition

$$h = h_{s,u} + h_f + r,$$

where r is the radical of h which is the Lie algebra of V^r. Suppose that $v \in ((g_{s,u} \cap h) \setminus h_{s,u})$. Since $h_{s,u} \in g_{s,u}$ we can assume that $v \in (h_f + r)$ after adding some vector w in $h_{s,u}$, if necessary. Let $v = v_f + v_r$ be the decomposition of v into the components in h_f and r. Assume that $v_f \neq 0$. Then the the closure of the 1-parameter subgroup $\exp tv$ in $\bar{H}$ is noncompact, since its projection to the quotient group $(\bar{H}/V^r) = \bar{H}_s$ is non-compact. Since $\bar{H}$ is a closed subgroup of $\bar{G}$, we conclude that the closure of $\exp tv$ in $\bar{G}$ is non-compact. On the other hand, the subgroup $\exp tv$ lies in the compact subgroup $\bar{G}_{s,u}$. Thus we arrive at a contradiction. Hence $v_f = 0$, so $v = v_r$. If $v_r \neq 0$, then the closure of the subgroup $\exp tv$ in $\bar{H}$ lies in V^r and it is also non-compact. Hence v_r also vanishes. This proves (3.4) and completes the proof of Proposition 3.3. □

Continuation of the proof of Proposition 3.2. Using Proposition 3.2 it suffces to consider only the transitive action of the compact, connected and simply connected group $\bar{G}_{s,u}$ on $X^7 = S^3 \times S^4$. Next we can assume that this action is almost effective, since otherwise the quotient $\bar{G}_{s,u}/N$ acts effectively and transitively on X^7, where N is the kernel of the action. The group $\bar{G}_{s,u}/N$ is compact, connected, and semi-simple. Its compact universal covering acts almost effectively on X^7.

Next we observe that the isotropy group of the action of $\bar{G}_{s,u}$ on X^7 is $\bar{H}_{s,u}$, since the isotropy group of this action must be connected. From now on we shall assume that $\bar{G}_{s,u} = \bar{H}_{s,u}$ admits a $\bar{G}_{s,u}$-invariant 3-form ω^3 of $\tilde{G}_2$-type on X^7.

Let $e = e \cdot \{H\}$ be a reference point on X^7. Denote by ρ the isotropy representation of $h_{s,u}$ on $T_e(\bar{G}/\bar{H}) = \mathbb{R}^7$. Since $h_{s,u}$ is semi-simple, the representation ρ is faithful, i.e. the kernel of ρ is empty. Clearly $\rho(h_{s,u})$ must be a sub-algebra of the Lie algebra $\tilde{g}_2$ of $\tilde{G}_2$, in particular the rank of $h_{s,u}$ is at most 2.

3.5. Lemma. *We have $h_{s,u} = su(2) \oplus su(2)$.*

Proof. Since ρ is faithful, we identify $h_{s,u}$ with its image in $\tilde{g}_2$. Clearly the algebra $h_{s,u}$ is also a subalgebra of a maximal compact Lie subalgebra in $\tilde{g}_2$, so $h_{s,u}$ is either $su(2)$ or $su(2) \times su(2)$.

Suppose that $h_{s,u} = su(2)$. Then $\dim \bar{G}_{s,u} = 10$ and therefore $\bar{G}_{s,u}$ must be a product of classical Lie groups. In particular we know that $\pi_4(\bar{G}_{s,u})$ is a finite group (see e.g. [8, §25.4].)

We also have $\pi_3(\bar{H}_{s,u}) = \mathbb{Z}$. Further we observe that the inclusion $\pi_3(\bar{H}_{s,u}) \to \pi_3(\bar{G}_{s,u})$ is injective, since the subgroup $\bar{H}_{s,u}$ realizes a non-trivial element in $H_3(\bar{G}_{s,u}, \mathbb{R})$. Now let us consider the homotopy exact sequence

$$\pi_4(\bar{G}) \to \pi_4(X^7) = \mathbb{Z}_2 \oplus \mathbb{Z} \to 0 = \; ker(\pi_3(\bar{H}) \to \pi_3(\bar{G})). \tag{3.6}$$

The exact sequence (3.6) implies that $\pi_4(\bar{G})$ contains a subgroup $\bar{Z}$ which is impossible. Hence $h_{s,u}$ cannot be $su(2)$. □

3.7. Lemma. *We have $g_{s,u} = su(2) \oplus so(5)$.*

Proof. Since $\dim g_{s,u} = 6 + 7 = 13$, it is easy to see that $g_{s,u}$ must be the product of classical compact Lie groups. Furthermore we notice that, since the dimension of $\bar{G}_{s,u}/\bar{H}_{s,u}$ is odd, the rank of $\bar{G}_{s,u}$ must be strictly greater than the rank of $\bar{H}_{s,u}$. Let $g^1_{s,u}$ be a simple component of $g_{s,u}$. Then the rank of $g^1_{s,u}$ is less than or equal to 2, otherwise the dimension of $g_{s,u}$ is greater than or equal to 15. Thus $g_{s,u}$ must be a sum of simple components of rank 1 or 2. Next, by dimension reason, $g_{s,u}$ cannot be a sum of only components of rank 1 and it cannot contain more than one simple component of rank 2. Looking at the table of simple Lie algebras we arrive at Lemma 3.7. □

3.8. Lemma. *The subalgebra $h_{s,u}$ lies in the component $so(5)$ of $g_{s,u}$.*

Proof. We notice that $\bar{G}_{s,u}$ has the same homotopy type as $\bar{G}$ and hence it is simply connected. Thus $\bar{G}_{s,u}$ is the product $SU(2) \times Spin(5)$. Analogously $\bar{H}_{s,u}$ must be $SU(2) \times SU(2)$. Denote by e_1 the composition of the embedding $e : h_{s,u} \to g_{s,u}$ with the projection $p : g_{s,u} \to su(2)$. To prove Lemma 3.8 it suffces to show that the kernel of e_1 is equal to $h_{s,u}$.

Suppose that the kernel of e_1 is not equal to $h_{s,u}$. Then this kernel must be one of the components $su(2)$ of $h_{s,u}$ since e_1 can not be injective. Denote this kernel by $su(2)^2$ and let $su(2)^1$ be the other component of $h_{s,u}$. The isomorphism $e_1 : su(2)^1 \to su(2)$ lifts to an isomorphism denoted by $\tilde{e}_1$ from the corresponding component $SU(2)^1$ of $\bar{H}_{s,u}$ to the component $SU(2)$ of $\bar{G}_{s,u}$. Denote by $\tilde{e}_2$ the homomorphism from $SU(2)^1 \times SU(2)^2$ to $Spin(5)$. Since $\tilde{e}_1(SU(2)^2) = Id$, the restriction of $\tilde{e}_2$ to $SU(2)^2$ is an embedding. We shall construct a map

$$I : \bar{G}_{s,u}/\bar{H}_{s,u} \to Spin(5)/\tilde{e}_2(SU(2)^2)$$

and show that I is a homeomorphism. For each point $(a \cdot b) \cdot \{\bar{H}_{s,u}\} \in \bar{G}_{s,u}/\bar{H}_{s,u}$, where $a \in SU(2)$ and $b \in Spin(5)$ we set

$$I((a,b) \cdot \{\bar{H}_{s,u}\}) = (b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \in Spin(5)/\tilde{e}_2(SU(2)^2). \tag{3.9}$$

The map (3.9) is well-defined, since for any $(\theta_1, \theta_2) \in SU(2)^1 \times SU(2)^2$ we have

$$\begin{aligned}
I(a \cdot \tilde{e}_1(\theta_1), b \cdot \tilde{e}_2(\theta_1, \theta_2))) \cdot \{\bar{H}_{s,u}\}) &= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a \cdot \tilde{e}_1(\theta_1))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\
&= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a) \cdot \theta_1)]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\
&= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\theta_1)]^{-1} \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\
&= (b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\
&= (b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot \tilde{e}_2(\theta_2)) \cdot \{\tilde{e}_2(SU(2)^2)\} = I((a,b) \cdot \{\bar{H}_{s,u}\}).
\end{aligned}$$

Substituting $a = 1$ into (3.9) we obtain that I is surjective. Now suppose that $I((a;b) \cdot \{\bar{H}_{s,u}\}) = I((a',b') \cdot \{H_{s,u}\})$. Then

$$b' \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a'))]^{-1} = b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot \tilde{e}_2(\theta_2) = b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1},$$

for some $\theta_2 \in SU(2)^2$. Hence

$$b' = b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a'))] = b \cdot \tilde{e}_2(\theta_2) \cdot \tilde{e}_2[(\tilde{e}_1^{-1}(a^{-1})) \cdot \tilde{e}_1^{-1}(a'))]. \tag{3.10}$$

Set $\theta_1 = \tilde{e}^{-1}(a^{-1} \cdot a')$. Then $a' = a \cdot \tilde{e}_1(\theta_1)$, and we get from (3.10)

$$b' = b \cdot \tilde{e}_2(\theta_2) \cdot \tilde{e}_2(\theta_1) = b \cdot \tilde{e}_2(\theta_1) \cdot \tilde{e}_2(\theta_2).$$

Hence I must be injective. Thus we have proved that I is a homeomorphism. Hence $\pi_4(Spin(5)/\tilde{e}_2(SU(2)^2) = \mathbb{Z}$. Now let us consider the homotopy exact sequence

$$\pi_4(Spin(5)) \xrightarrow{j} \pi_4(Spin(5)/\tilde{e}_2(SU(2)^2)) \to \pi_3(\tilde{e}_2(SU(2)^2) \xrightarrow{i} \pi_3(Spin(5)).$$

Clearly i is injective. Hence j must be surjective. But $\pi_4(Spin(5))$ is a finite group. Therefore $\pi_4(Spin(5)/\tilde{e}_2(SU(2)^2)$ is also a finite group. Thus we arrive at a contradiction, since $\pi_4(X^7)$ contains an infinite subgroup. This proves Lemma 3.8. $\square$

Completion of the proof of Proposition 3.2. From Lemma 3.8 it follows that the restriction of the isotropy action of $h_{s,u}$ on each component $su(2)^i$ contains the sum of three trivial representations. Now look at the table of the irreducible 7-dimensional representation π_1 of g_2 (see e.g. [13, table 1]) we know that the weights of this representation are $\pm\varepsilon_i, 0$. Here $i = \overline{1,3}$ and $\varepsilon_i - \varepsilon_j\ \pm\varepsilon_i$ are the roots of g_2. The complexification of the maximal compact algebra $so(4)$ in g_2 is the direct sum $so(4)_{\otimes\mathbb{C}} =< h_{\varepsilon_1}, e_{\varepsilon_1}, e_{-\varepsilon_1} >_{\otimes\mathbb{C}} \oplus < h_{\varepsilon_2-\varepsilon_3}, e_{\varepsilon_2-\varepsilon_3}, e_{\varepsilon_3-\varepsilon_2} >_{\otimes\mathbb{C}}$. Hence the restriction of the π_1 to $sl(2) =< h_{\varepsilon_1}, e_{\varepsilon_1}, e_{-\varepsilon_1} >_{\otimes\mathbb{C}}$ is the sume of two irreducible components of dimension 2 and the adjoint representation of dimension 3. Thus it has no trivial component. We arrive at a contradiction. This proves Proposition 3.2. $\square$

4 Appendix. A necessary and sufficient condition for a closed 7-manifold to admit a 3-form of $\tilde{G}_2$-type.

A.1. Theorem. *Suppose that M^7 is a compact 7-manifold. Then M^7 admits a 3-form of $\tilde{G}_2$-type, if and only if M^7 is orientable and spinnable. Equivalently the first and the second Stiefel-Whitney classes of M^7 vanish.*

Proof. If M^7 admits a $\tilde{G}_2$-structure, it admits also a G_2-structure, since a maximal compact group of $\tilde{G}_2$ is also a compact subgroup of the group G_2. Applying the Gray criterion for the existence of a G_2-structure [9] we obtain the "only if" statement. Now let us prove the "if " part. By a result of Dupont [4] any compact orientable 7-manifold admits three linearly independent vector fields. Hence M^7 admits an $SO(4)$-structure. In particular M^7 admits an $SO(3)\times SO(4)$-structure. Denote by $Spin(3,4)$ the Lie subgroup in $Spin(7,\mathbb{C})$ whose Lie algebra is $so(3,4)$. It is known (see e.g. [2 Theorem5]) that $\pi_1(Spin(3,4)) = \mathbb{Z}_2$. Denote by $(SO(3)\times SO(4))^*$ the connected subgroup in $Spin(3,4)$ whose Lie algebra is $so(3)\times so(4)$. Clearly $(SO(3)\times SO(4))^*$ is a maximal compact Lie subgroup of $Spin(3,4)$. Taking into account the isomorphism $Spin(7)/(SO(3)\times SO(4))^* = SO(7)/(SO(3)\times SO(4))$ we conclude that M^7 admits an $(SO(3)\times SO(4))^*$-structure and hence a $Spin(3,4)$-structure. Now we shall prove that the $Spin(3,4)$-structure on M^7 is reduced to a $\tilde{G}_2$-structure. It is easy to see that the quotient $Spin(3,4)/\tilde{G}_2$ is the pseudo-sphere $S^7(4,4)$ in the space $\mathbb{R}e_0 \times \mathbb{R}^7$ of the spinor representation of $Spin(3,4)$. This pseudo-sphere bundle over M^7 admits a section e_0 . Hence M^7 admits a $\tilde{G}_2$-structure. $\square$

Using Theorem A.1 and the Gromov h-principle (see e.g. [5]) for open 3-forms of $\tilde{G}_2$-type we can get a lot examples of open 7-manifolds admitting a closed 3-form of $\tilde{G}_2$-type.

Acknowledgement. I am thankful to Jiri Vanzura and Jarolim Bures for their introducing me to the field of multi-symplectic 3-forms. The main part of this paper has been written at IHES. This paper is partially supported by contract RITA-CT-2004-505493 and by grant of ASCR Nr IAA100190701.

References

[1] J. Bures and J. Vanzura, *Multisymplectic forms of degree three in dimension seven*, suppl. Rend. del Circolo Math. Palermo, serie II, 71, (2003), 73-91.

[2] R. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. (2), 126 (1987), 525-576.

[3] R. Cleyton and S. Ivanov, *On the geometry of closed G_2-structures*, arXiv:math.DG/0306362.

[4] J. Dupont, *K-theory obstructions to the existence of vector fields*, Acta Math. 133 (1974), 67–80.

[5] Y. Eliashberg and N. Mishachev, *Introduction to the h-Principle*, AMS 2002.

[6] M. Fernandez, *An example of compact calibrated manifold associated with exceptional Lie group G_2*, J.D.G. 26 (1987), 367-370.

[7] M. Fernandez and A. Gray, *Riemannian manifold with structure group G_2*, Ann. Math. Pura Appl. 3, 132 (1982), 19-45.

[8] A.T. Fomenko and D. B. Fuks, A course in homotopic topology, Moscow nauka 1989.

[9] A. Gray, *Vector cross products on manifolds*, TAMS 141, (1969), 465-504, (Errata in TAMS 148 (1970), 625).

[10] N. Hitchin, *Stable forms and special metrics*, Contemporean math., (2001), 288, 70-89.

[11] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2, I, II*, J.D.G. 43(1996), 291-328 and 329-375.

[12] E.N. Vinberg and A.L.Onishik, Seminar on Lie groups and algebraic groups, (in Russian), Moscow Nauka 1988.

address: Mathematical Institute of ASCR, Zitna 25, Praha 1, CZ-11567 Czech Republic, email: hvle@math.cas.cz